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Motion of an axisymmetric body in a rotating stratified fluid confined between two parallel planes

By D. V. KRISHNA AND L. V. SARMA
Department of Mathematics,
Indian Institute of Technology, Madras–36

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We consider here the flow due to the oscillation of a slender oblate spheroid in a non-homogeneous, rotating fluid confined between two parallel planes which are perpendicular to the (vertical) axis of rotation. The direction of oscillation of the spheroid is perpendicular to the axis of rotation. By solving a set of dual integrals the steady-state solution is obtained in the two cases when the plates are at an infinite distance from the body and when they are at a large but finite distance. The singular or discontinuous surfaces observed in the case of homogeneous fluid are absent here. Also, the steady-state velocity is no longer independent of the distance along the axis of rotation. The velocity has now a vertical gradient, an important feature in the case of stratified fluid. It is also found that the presence of the plane boundaries increases the force on the body.

1. Introduction

When an axisymmetric body moves along its axis of revolution in a rotating fluid the disturbance created by it has a singular character on the circumscribing cylinder $\mathscr{C}$ and the fluid inside $\mathscr{C}$ is pushed along the body while outside $\mathscr{C}$ the flow tends to be steady and two-dimensional. This phenomenon was studied experimentally by Taylor (1923) whose observations were confirmed theoretically by, for instance, Grace (1927), Stewartson (1953), etc. When the motion is perpendicular to the axis of symmetry, Taylor (1923) observed similar phenomena inside the circumscribing cylinder $\mathscr{C}$ but outside it the motion was highly asymmetric. Recently, Stewartson (1967) has studied this problem when the rotating fluid is bounded by parallel planes. Starting with an initial value problem, he deduces that the ultimate flow ($t \rightarrow \infty$) is two-dimensional outside $\mathscr{C}$, whereas inside $\mathscr{C}$ the fluid is stagnant in the rotating frame. Also he shows that the discontinuity on the cylinder $\mathscr{C}$ can be thought of as an arbitrary thin shear layer providing smooth transition between the fluid exterior and interior of $\mathscr{C}$. Partial explanations are given for the observed asymmetry outside $\mathscr{C}$.

The problem of three-dimensional disturbances caused by an axisymmetric body when it moves in a non-homogeneous fluid rotating with a constant angular velocity, has received some attention in recent times, as it involves certain features different from those found in the case of homogeneous fluid. In an earlier work (Krishna & Sarma 1969*a*) the authors studied the axisymmetric flow created by an oscillating body in an unbounded stratified fluid, subject to constant rotation about the axis of revolution of the body. It is found that it is not possible to obtain steady-state solutions on the basis of linearized theory, though no singular or discontinuous surfaces arise in the limit as the frequency of oscillation tends to zero. On the other hand, when the non-homogeneous rotating fluid is bounded externally by an infinite cylinder, the steady-state solutions exist. For example, a solution has been obtained in the case of a disk moving along the axis by using Oseen-type steady-state equations (Krishna & Sarma 1969*b*).

In this paper, we have studied the flow created by oscillations of a spheroid in a stratified fluid rotating with a constant angular velocity between two horizontal plane boundaries such that relative to the rotating axis (which coincides with the axis of revolution) the motion is in a perpendicular direction to it. The body is assumed to be slender in order to satisfy the boundary condition on the disk. Steady-state solutions are obtained in the two cases when the plane boundaries are infinite and when they are finite but at large distances. Forces acting on the body are also calculated in both cases. In contrast to homogeneous rotating fluid, here there exist no singular or discontinuous surfaces. Also from the expressions for u, v in the limiting case, we note that the steady-state velocity is no longer independent of the distance measured along the axis of rotation but gives rise to the vertical gradient of horizontal velocity thus violating the Taylor–Proudman constraint of homogenous fluids (Barcilon & Pedlosky 1967). It is also to be noted that the influence of plane boundaries increases the force acting on the spheroid in the transverse direction.

2. Governing equations

Consider a set of rectangular Cartesion axes (Ox, Oy, Oz), Oz being measured in the vertical direction, opposing gravity. The equations of motion of an inviscid fluid with respect to a frame of reference rotating with a constant angular velocity about z axis can be written in vector form as (Yih 1965)

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\,.\,\nabla)\mathbf{v}\right) = -\nabla p+\rho\mathbf{X}-2\rho\mathbf{\Omega}\times\mathbf{v}-\rho\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}),$$

in which $\mathbf{v}$ is the velocity vector, $\mathbf{r}$ the co-ordinate vector and $\mathbf{X}$ represents the body force. In the case of stratified incompressible fluid, gravity being the only body force, the components X_x and X_y are zero, and the governing equations are

$$\rho\frac{du}{dt} = -\frac{\partial P}{\partial n}+2\rho\Omega v+\rho\Omega^2 x, \tag{2.1}$$

$$\rho\frac{dv}{dt} = -\frac{\partial P}{\partial y}-2\rho\Omega u+\rho\Omega^2 y, \tag{2.2}$$

$$\rho\frac{dw}{dt} = -\frac{\partial P}{\partial z}-\rho g, \tag{2.3}$$

$$\left(\frac{d}{dt} = \frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right),$$

where u, v, w denote the velocity components along x, y, z, directions respectively.

Since the fluid is incompressible

$$d\rho/dt = 0. \tag{2.4}$$

The equation of continuity reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0. \tag{2.5}$$

The stratification in the undisturbed state is taken to be linear and a function of z alone so that the initial density ρ_0 is equal to $\rho_0'(1-\beta z)$ where β and ρ_0' are constants. Let P_0 be the corresponding pressure.

The fluid is assumed to be confined within two parallel planes each perpendicular to the axis of rotation (but otherwise unbounded). Consider an oblate spheroid whose axis of revolution coincides with the axis of rotation, oscillating along the Ox direction with a velocity $Ue^{i\sigma t}$, such that relative to the rotating fluid, the motion is in a straight line perpendicular to the axis. Choosing the origin to be at the centre, the equation of the spheroid can be taken as $(x^2+y^2)/a^2+z^2/b^2 = 1$.

Let the components of the fluid velocity in the perturbed state, relative to instantaneously fixed axes at the centre of the body, be u, v, w. Let the subsequent density and pressure be ρ and P, respectively. Substituting u, v, w, P_0+P, $\rho_0+\rho$ in the equations (2.1)–(2.4) and using the Boussinesq approximation, the linearized equations of motion can be written as

$$\rho_0'\left(\frac{\partial u}{\partial t}-2\Omega v\right) = -\frac{\partial P}{\partial x}, \tag{2.6}$$

$$\rho_0'\left(\frac{\partial v}{\partial t}+2\Omega u\right) = -\frac{\partial P}{\partial y}, \tag{2.7}$$

$$\rho_0'\frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z}-\rho g, \tag{2.8}$$

$$\frac{\partial \rho}{\partial t}+w\frac{\partial \rho_0}{\partial z} = 0, \tag{2.9}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0. \tag{2.10}$$

The boundary condition on the spheroid is that

$$\frac{ux+vy}{a^2}+\frac{wz}{b^2} = \frac{Ux}{a^2}e^{i\sigma t} \quad \text{on} \quad \frac{x^2+y^2}{a^2}+\frac{z^2}{b^2} = 1, \tag{2.11}$$

and the condition on the planes is

$$w = 0 \quad \text{on} \quad |z| = h. \tag{2.12}$$

Taking oscillations for the variables as $P = P'e^{i\sigma t}$; $u = u'e^{i\sigma t}$ etc. and eliminating u', v' w' from (2.6)–(2.8) using (2.9) and (2.10), the governing equation in terms of P' is

$$\frac{\partial^2 P'}{\partial x^2}+\frac{\partial^2 P'}{\partial y^2}+\frac{4\Omega^2-\sigma^2}{\beta g-\sigma^2}\frac{\partial^2 P'}{\partial z^2} = 0, \tag{2.13}$$

and u', v', w' expressed in terms of P' are

$$\left.\begin{aligned} u' &= \frac{-i\sigma}{\rho'(4\Omega^2-\sigma^2)}\left[\frac{\partial P'}{\partial x}+\frac{2\Omega}{i\sigma}\frac{\partial P'}{\partial y}\right], \quad v' = \frac{-i\sigma}{\rho_0'(4\Omega^2-\sigma^2)}\left[\frac{\partial P'}{\partial y}-\frac{2\Omega}{i\sigma}\frac{\partial P'}{\partial x}\right], \\ w' &= \frac{-i\sigma}{\rho_0'(\beta g-\sigma^2)}\frac{\partial P'}{\partial z}. \end{aligned}\right\} \tag{2.14}$$

Thus (2.11) reduces to

$$\left(x\frac{\partial P'}{\partial x}+y\frac{\partial P'}{\partial y}\right)+\frac{2\Omega}{i\sigma}\left(x\frac{\partial P'}{\partial y}-y\frac{\partial P'}{\partial x}\right)-\frac{a^2(4\Omega^2-\sigma^2)}{b^2(\beta g-\sigma^2)}z\frac{\partial P'}{\partial z}$$
$$=\frac{i\rho_0'U(4\Omega^2-\sigma^2)x}{\sigma} \quad \text{on} \quad \frac{x^2+y^2}{a^2}+\frac{y^2}{b^2}=1. \tag{2.15}$$

Introducing cylindrical polars (r', θ)

$$x = r'\cos\theta, \quad y = r'\sin\theta,$$

(2.13) transforms to

$$\frac{\partial^2 P'}{\partial r'^2}+\frac{1}{r'}\frac{\partial P'}{\partial r'}+\frac{1}{r'^2}\frac{\partial^2 P'}{\partial \theta^2}+\frac{4\Omega^2-\sigma^2}{\beta g-\sigma^2}\frac{\partial^2 P'}{\partial z^2}=0,$$

with the condition (2.15) as

$$r'\frac{\partial P'}{\partial r'}+\frac{2\Omega}{i\sigma}\frac{\partial P'}{\partial \theta}+\frac{(a^2-r'^2)^{\frac{1}{2}}}{\lambda}\frac{\partial P'}{\partial z}$$
$$=\frac{i\rho_0'U(4\Omega^2-\sigma^2)\,r'\cos\theta}{\sigma}, \quad \text{when} \quad r'^2+z^2/\lambda^2=a^2 \quad (\lambda=b/a).$$

Using the transformation

$$r'=ra, \quad z=a\bar{z}\left(\frac{4\Omega^2-\sigma^2}{\beta g-\sigma^2}\right)^{\frac{1}{2}},$$

the above equations can be written as

$$\frac{\partial^2 P'}{\partial r^2}+\frac{1}{r}\frac{\partial P'}{\partial r}+\frac{1}{r^2}\frac{\partial^2 P'}{\partial \theta^2}+\frac{\partial^2 P'}{\partial \bar{z}^2}=0,$$

$$r\frac{\partial P'}{\partial r}+\frac{2\Omega}{i\sigma}\frac{\partial P'}{\partial \theta}+\frac{(1-r^2)^{\frac{1}{2}}}{\lambda}\left(\frac{4\Omega^2-\sigma^2}{\beta g-\sigma^2}\right)^{\frac{1}{2}}\frac{\partial P'}{\partial \bar{z}}$$
$$=\frac{i\rho_0'Ua(4\Omega^2-\sigma^2)\,r\cos\theta}{\sigma} \quad \text{when} \quad r^2+\frac{\bar{z}^2}{\lambda^2}\left(\frac{4\Omega^2-\sigma^2}{\beta g-\sigma^2}\right)=1.$$

From (2.12)

$$\frac{\partial P'}{\partial \bar{z}}=0 \quad \text{on} \quad |\bar{z}|=\frac{h}{a}\left(\frac{\beta g-\sigma^2}{4\Omega^2-\sigma^2}\right)^{\frac{1}{2}}=H \quad \text{(say)}.$$

3. Solution of the problem

In view of the condition on the body we take

$$P'=\mathrm{Re}[Q\,e^{i\theta}].$$

Whence Q satisfies

$$\frac{\partial^2 Q}{\partial r^2}+\frac{1}{r}\frac{\partial Q}{\partial r}-\frac{Q}{r^2}+\frac{\partial^2 Q}{\partial \bar{z}^2}=0 \tag{3.1}$$

with the condition

$$r\frac{\partial Q}{\partial r}+\frac{2\Omega}{\sigma}Q+(1-r^2)^{\frac{1}{2}}\lambda'\frac{\partial Q}{\partial \bar{z}} = \frac{i\rho_0' Ua(u\Omega^2-\sigma^2)}{\sigma}r, \tag{3.2}$$

on

$$r^2+\frac{\bar{z}^2}{\lambda^2}\left(\frac{u\Omega^2-\sigma^2}{\beta g-\sigma^2}\right) = 1 \tag{3.3}$$

and

$$\frac{\partial Q}{\partial \bar{z}} = 0 \quad \text{on} \quad |\bar{z}| = H.$$

$$\left(\lambda' = \frac{1}{\lambda}\left(\frac{u\Omega^2-\sigma^2}{\beta g-\sigma^2}\right)^{\frac{1}{2}}\right).$$

The general solution of the equation (3.1) in the region $0 < \bar{z} < H$ satisfying the condition on the upper plane $\bar{z} = H$ can be expressed as

$$Q = \int_0^\infty J_1(\kappa r)\frac{\cosh\kappa(H-\bar{z})}{\sinh\kappa H}A(\kappa)\,d\kappa. \tag{3.4}$$

From the nature of the solution, we see that the flow is symmetric about the x, y plane and hence (2.14) gives

$$\frac{\partial Q}{\partial \bar{z}} = 0 \quad \text{on} \quad \bar{z} = 0 \quad (r > 1). \tag{3.5}$$

(Thus the above solution (3.4) can be extended to the region $\bar{z} < 0$ by symmetry.)

Assuming the body to be slender ($\lambda \ll 1$) we can replace (3.3) by $\bar{z} = 0$, thus satisfying the boundary condition (3.2) on $\bar{z} = 0$ $(r < 1)$.

Hence the conditions (3.5) and (3.2) give

$$\int_0^\infty \kappa A(\kappa)J_1(\kappa r)\,d\kappa = 0 \quad (r > 1), \tag{3.6}$$

$$r\int_0^\infty A(\kappa)J_1'(\kappa r)\coth\kappa H\,d\kappa+\frac{2\Omega}{\sigma}\int_0^\infty A(\kappa)J_1(\kappa r)\coth\kappa H\,d\kappa$$
$$+\lambda'(1-r^2)^{\frac{1}{2}}\int_0^\infty \kappa A(\kappa)J_1(\kappa r)\,d\kappa = \frac{i\rho_0' Ua(4\Omega^2-\sigma^2)}{\sigma}r \quad (r < 1) \tag{3.7}$$

($'$ denotes the differentiation with respect to κr).

Case (i)

When the plates are situated at a very large distance ($h \gg a$) such that $H \simeq \infty$, we take $\coth\kappa H \simeq 1$ and the equation (3.7) of dual integral equations reduces to

$$r\int_0^\infty A(\kappa)J_1'(\kappa r)\,d\kappa+\frac{2\Omega}{\sigma}\int_0^\infty A(\kappa)J_1(\kappa r)\,d\kappa+\lambda'(1-r^2)^{\frac{1}{2}}\int_0^\infty \kappa A(\kappa)J_1(\kappa r)\,d\kappa$$
$$= \frac{i\rho_0' Ua(4\Omega^2-\sigma^2)}{\sigma}r \quad (r < 1). \tag{3.8}$$

The appropriate solution of (3.6) can be (Stewartson 1967) taken as

$$A(\kappa) = B\left(\frac{2\pi}{\kappa}\right)^{\frac{1}{2}}J_{\frac{3}{2}}(\kappa).$$

Substituting this into equation (3.8) and simplifying, we get

$$B = \frac{2i\rho_0' Ua(4\Omega^2-\sigma^2)}{\pi(2\Omega+\sigma)+4\sigma\lambda'}.$$

Hence

$$P = \mathrm{Re}\left[\frac{2\sqrt{(2\pi)}\, i\rho_0' Ua(4\Omega^2-\sigma^2)}{\pi(2\Omega+\sigma)+4\sigma\lambda^1} e^{i(\sigma t+\theta)} \int_0^\infty \kappa^{-\frac{1}{2}} J_{\frac{3}{2}}(\kappa)\, J_1(\kappa r)\, e^{-\kappa z}\, d\kappa\right]. \quad (3.9)$$

When Ω, $\beta \neq 0$. (2Ω, $\sqrt{(\beta g)}$ are taken to be greater than σ.) On plane $\bar{z} = 0$, i.e. (x, y) plane,

$$P = \frac{-\pi\rho_0' Ua\lambda(4\Omega^2-\sigma^2)(\beta g-\sigma^2)^{\frac{1}{2}}\, r\sin(\sigma t+\theta)}{\pi\lambda(2\Omega+\sigma)(\beta g-\sigma^2)^{\frac{1}{2}}+4\sigma(4\Omega^2-\sigma^2)^{\frac{1}{2}}} \quad (r < 1)$$

$$= \frac{-2\rho_0' Ua\lambda(4\Omega^2-\sigma^2)(\beta g-\sigma^2)^{\frac{1}{2}}}{\pi\lambda(2\Omega+\sigma)(\beta g-\sigma^2)^{\frac{1}{2}}+4\sigma(4\Omega^2-\sigma^2)^{\frac{1}{2}}}\, r\left(\sin^{-1}\frac{1}{r}-\frac{(r^2-1)^{\frac{1}{2}}}{r}\right)\sin(\sigma t+\theta) \quad (r > 1).$$

At any finite distance $\bar{z}(\neq 0)$

$$P = \frac{-\sqrt{\pi}\rho_0' Ua\lambda(4\Omega^2-\sigma^2)(\beta g-\sigma^2)^{\frac{1}{2}}\, r\sin(\sigma t+\theta)}{[2\pi\lambda(2\Omega+\sigma)(\beta g-\sigma^2)^{\frac{1}{2}}+4\sigma(4\Omega^2-\sigma^2)^{\frac{1}{2}}]\,\bar{z}^3} \times \sum_{\kappa=0}^{\infty}\frac{\Gamma(2\kappa+3)}{\kappa!\,\Gamma(\kappa+\frac{5}{2})}\left(-\frac{1}{4\bar{z}^2}\right)^{\kappa} {}_2F_1\left(\begin{matrix}-\kappa, \frac{3}{2}, \kappa \\ 2\end{matrix}\; n^2\right).$$

Hence in the limit as $\sigma \to 0$, on $(x; y)$ plane

$$P \to -2\rho_0' U\Omega y \quad \text{if} \quad x^2+y^2 \leqslant a^2$$

$$\to -\frac{4\rho_0' U\Omega y}{\pi}\left(\sin^{-1}\frac{a}{(x^2+y^2)^{\frac{1}{2}}}-\frac{a(x^2+y^2-a^2)^{\frac{1}{2}}}{x^2+y^2}\right) \quad \text{if} \quad x^2+y^2 \geqslant a^2.$$

At any point inside the fluid ($z \neq 0$)

$$P \to -\frac{8\rho_0' U\Omega y}{3\pi}\left\{1-\frac{6}{5}\left(\frac{a}{z}\right)^2\left(\frac{4\Omega^2}{\beta g}\right)\left(1+\frac{5(x^2+y^2)}{4}\right)+\ldots\right\}\left(\frac{4\Omega^2 a^2}{\beta g z^2}\right)^{\frac{3}{2}}.$$

Thus we see that the flow tends to a steady state, in the limiting case $\sigma \to 0$. Also it can be shown that in the limit as σ approaches zero, u, v are finite whereas w tends to zero.

The forces on the spheroid in this case (calculated up to the $O(\lambda^3)$) are

$$F_x = 0, \quad \underset{\text{(hydrostatic thrust)}}{F_z = +\frac{2\pi\rho_0' a^3\lambda}{3}\left(\frac{\beta a^2 r^2}{5}-g\right)},$$

$$F_y = 4\rho_0' U\Omega a^3\lambda\left\{\frac{2\pi}{3}-\frac{4\lambda}{3}\left(\frac{\beta g}{4\Omega^2}\right)^{\frac{1}{2}}+O(\lambda^2)\right\}.$$

When $\beta = 0$; $\Omega \neq 0$. This case has been discussed by Stewartson (1967), who tackles it as an initial-value problem for the case of a sphere. When the plane boundaries are at very large distances the general expression for P is given by

(3.9). Since, when $\beta = 0$; $z = 0$, in the expression (3.9) putting $z = 0$ and taking the limit $\sigma \to 0$ we get

$$P = \frac{2\pi\rho_0' U\lambda\Omega(4x-\pi\lambda y)}{16+\pi^2\lambda^2} \quad \text{if} \quad x^2+y^2 \leqslant a^2$$

and

$$P = \frac{4\rho_0' U\Omega\lambda}{16+\pi^2\lambda^2}(4x-\pi\lambda y)\left(\sin^{-1}\frac{a}{(x^2+y^2)^{\frac{1}{2}}} - \frac{a(x^2+y^2-a^2)^{\frac{1}{2}}}{x^2+y^2}\right) \quad \text{if} \quad x^2+y^2 \geqslant a^2.$$

These expressions coincide with those given by Stewartson (1967) when $\lambda = 1$.

When $\Omega = 0$; $\beta \neq 0$. In this case, H, z, both tend to infinity as σ approaches zero and hence from (3.9) $P \to 0$ as $\sigma \to 0$. This particular case has been studied in detail in another paper (Krishna & Sarma 1969*c*).

Case (ii)

When the distance h is large but finite, the solution can be generalized using the expansion of $\coth \kappa H$ as

$$\coth \kappa H = 1 + 2\sum_{n=1}^{\infty} e^{-2n\kappa H}.$$

The main interest here is to study the effect of plane boundaries on the force acting on the spheroid. Replacing $\coth \kappa H$ by the above exponential series, (3.7) can be expressed as

$$r\int_0^\infty \kappa A(\kappa)\,J_0(\kappa r)\,d\kappa + \left(\frac{2\Omega}{\sigma}-1\right)\int_0^\infty A(\kappa)\,J_1(\kappa r)\,du + \lambda'(1-r^2)^{\frac{1}{2}}\int_0^\infty \kappa A(\kappa)\,J_1(\kappa r)\,d\kappa$$

$$= \frac{i\rho_0' U a r(4\Omega^2-\sigma^2)}{\sigma} - 2\sum_n\left\{r\int_0^\infty \kappa A(\kappa)\,J_0(\kappa r)\,e^{-2n\kappa H}\,d\kappa + \left(\frac{2\Omega}{\sigma}-1\right)\int_0^\infty A(\kappa)\,J_1(\kappa r)\,e^{-2n\kappa H}\,d\kappa\right\} \quad \text{when} \quad r < 1. \qquad (3.10)$$

We choose the solution of (3.6) as (Stewartson 1967)

$$A(\kappa) = \left(\frac{2\pi}{\kappa}\right)^{\frac{1}{2}}\left[B_1 J_{\frac{3}{2}}(\kappa) + \frac{B_2}{\kappa}J_{\frac{5}{2}}(\kappa) + \dots\right].$$

Wherein each term of the series satisfies the equation identically, and B_1, B_2 etc. are constants to be determined. Substituting the above expression for $A(\kappa)$ in (3.10) and solving for B_1, B_2 we find

$$B_1 = \frac{2i\rho_0' U a(4\Omega^2-\sigma^2)}{(2\Omega+\sigma)\pi+4\sigma\lambda'}\left\{1 - \frac{(2\Omega+\sigma)\zeta(3)}{3[(2\Omega+\sigma)\pi+4\sigma\lambda']H^3} + \left(\frac{2\Omega+5\sigma}{6\Omega\pi+(9\pi+32\lambda')\sigma} + \frac{2\Omega+\sigma}{60[(2\Omega+\sigma)\pi+4\sigma\lambda']}\right)\frac{\zeta(5)}{H^5} - \frac{4(2\Omega+5\sigma)(2\Omega+\sigma)\zeta(5)\zeta(3)}{5[6\Omega\pi+(9\pi+32\lambda')\sigma][(2\Omega+\sigma)\pi+4\sigma\lambda']H^8} + O(H^{-11})\right\},$$

$$B_2 = \frac{-2i\rho_0' U a(4\Omega^2-\sigma^2)(2\Omega+5\sigma)\zeta(5)}{[(2\Omega+\sigma)\pi+4\sigma\lambda'][6\Omega\pi+(9\pi+32\lambda')\sigma} \times\left\{\frac{1}{H^5} - \frac{(2\Omega+\sigma)\zeta(3)}{3[(2\Omega+\sigma)\pi+4\sigma\lambda']H^8} + O(H^{-11})\right\},$$

neglecting terms of $O(H^{-11})$. Here $\zeta(\alpha)$ is the Riemann-zeta function given by

$$\zeta(\alpha) = \sum_{n=1}^{\infty} 1/\eta^{\alpha}.$$

Thus we note that the order of the constants B_2 etc. goes on decreasing as inverse powers of H and hence are negligible when H is sufficiently large.

At any point in the region $0 < \bar{z} < H$

$$P = \mathrm{Re}\left[e^{i(\sigma t+\theta)}\left\{\frac{\sqrt{\pi}B_1 r}{4z^3}\sum_{n=0}^{\infty}\frac{\Gamma(2n+3)}{m!\,\Gamma(n+\frac{5}{2})}\left(-\frac{1}{4z^2}\right)^n {}_2F_1\left(\begin{matrix}-n, & -\frac{3}{2}-n \\ & 2\end{matrix}\; r^2\right)\right.\right.$$

$$+\frac{\sqrt{\pi}B_2 r}{8z^3}\sum_{n=0}^{\infty}\frac{\Gamma(2n+3)}{n!\,\Gamma(n+\frac{7}{2})}\left(-\frac{1}{4z^2}\right)^n {}_2F_1\left(\begin{matrix}-n, & -\frac{5}{2}-n \\ & 2\end{matrix}\; r^2\right)$$

$$+2r\left\langle\sum_{n=1}^{\infty}\frac{1}{(2nH-z)^3}\left(\frac{B_1}{3}\left(1-\frac{6}{5(2nH-z)^2}\right)+\frac{B_2}{15}\left(1-\frac{6}{7(2nH-z)^2}\right)\right)\right.$$

$$\left.+\sum_{n=1}^{\infty}\frac{1}{(2nH+z)^3}\left(\frac{B_1}{3}\left(1-\frac{6}{5(2nH+z)^2}\right)+\frac{B_2}{15}\left(1-\frac{6}{7(2nH+z)^2}\right)\right)\right\rangle$$

$$\left.\left.-r^3\left(B_1+\frac{B_2}{5}\right)\sum_{n=1}^{\infty}\left(\frac{1}{(2nH-z)^5}+\frac{1}{(2nH+z)^5}\right)\right\}\right].$$

At any point on (x, y) plane

$$P = \mathrm{Re}\left[e^{i(\sigma t+\theta)}\left\{r\left\langle B_1\left[\frac{\pi}{2}+\left(\zeta(3)-\frac{3\zeta(5)}{10H^2}\right)\frac{1}{6H^3}\right]+B_2\left[\frac{\pi}{4}+\left(\zeta(3)-\frac{3\zeta(5)}{14H^2}\right)\frac{1}{30H^3}\right]\right\rangle\right.\right.$$

$$\left.\left.-r^3\left\langle\frac{3\pi B_2}{16}+\left(B_1+\frac{B_2}{5}\right)\frac{\zeta(5)}{16H^2}\right\rangle\right\}\right] \quad \text{if} \quad r \leqslant 1,$$

$$= \mathrm{Re}\left[e^{i(\sigma t+\theta)}\left\{r\left\langle B_1\left[\sin^{-1}\frac{1}{r}-\frac{(r^2-1)^{\frac{1}{2}}}{r}+\left(\zeta(3)-\frac{3\zeta(5)}{10H^2}\right)\frac{1}{6H^3}\right]\right.\right.\right.$$

$$\left.+B_2\left[\frac{2}{15r^3}\,{}_2F_1\left(\begin{matrix}\frac{3}{2}, \frac{1}{2}, \\ \frac{7}{2}\end{matrix}\; r^{-2}\right)+\left(\zeta(3)-\frac{3\zeta(5)}{14H^2}\right)\frac{1}{30H^3}\right]\right\rangle$$

$$\left.\left.-\frac{r^3\zeta(5)}{16H^5}\left(B_1+\frac{B_2}{5}\right)\right\}\right] \quad \text{if} \quad r \geqslant 1.$$

At points very near the plate,

$$P = \mathrm{Re}\left[e^{i(\sigma t+\theta)}\left\{r\left\langle\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3H^3}\left(\frac{4B_1}{3}\left(1-\frac{6}{5(2n+1)^2H^2}\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\frac{4B_2}{15}\left(1-\frac{6}{7(2n+1)^2H^2}\right)\right)\right\rangle-\frac{2r^3}{H^5}\left(B_1+\frac{B_2}{5}\right)\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\right\}\right].$$

In the limit $\sigma \to 0$, Z goes to $z/a(\beta g/4r^2)^{\frac{1}{2}}$ and H goes to $h/a(\beta g/4r^2)^{\frac{1}{2}}$ and

$$B_1 \to \frac{4i\rho_0' a U\Omega}{\pi}\left\{1-\frac{\zeta(3)}{3\pi H^3}+\frac{7\zeta(5)}{20\pi H^5}-\frac{4\zeta(5)\zeta(3)}{15\pi^2H^8}+O(H^{-11})\right\},$$

$$B_2 \to \frac{-4i\rho_0' a U\Omega\zeta(5)}{3\pi^2H^5}\left\{1-\frac{\zeta(3)}{3\pi H^3}+O(H^{-6})\right\}.$$

Hence P tends to a finite limit as σ approaches zero.

Using (2.14) we find

$$u \to \frac{\sqrt{\pi}}{4}\left(\frac{4\Omega^2}{\beta g}\right)^{\frac{3}{2}}\left(\frac{a}{z}\right)^3\left[(B_1)_{\sigma=0}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(2n+3)}{n!\,\Gamma(n+\frac{5}{2})}\left(\frac{-\Omega^2a^2}{\beta g z^2}\right)^n {}_2F_1\left(-n, \frac{-\frac{3}{2}-n}{2}; x^2+y^2\right)\right\}\right.$$

$$+2y^2\sum_{n=0}^{\infty}\frac{\Gamma(2n+3)}{n!\,\Gamma(n+\frac{5}{2})}\left(\frac{-\Omega^2a^2}{\beta g z^2}\right)^n {}_2F_1'\left(-n, \frac{-\frac{3}{2}-n}{2}; x^2+y^2\right)\Bigg\}$$

$$+\frac{(B_2)_{\sigma=0}}{2}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(2n+3)}{n!\,\Gamma(n+\frac{7}{2})}\left(\frac{-\Omega^2a^2}{\beta g z^2}\right)^n {}_2F_1\left(-n, \frac{-\frac{5}{2}-n}{2}; x^2+y^2\right)\right.$$

$$\left.\left.+2y^2\sum_{n=0}^{\infty}\frac{\Gamma(2n+3)}{n!\,\Gamma(n+\frac{7}{2})}\left(\frac{-\Omega^2a^2}{\beta g z^2}\right)^n {}_2F_1'\left(-n, \frac{-\frac{5}{2}-n}{2}, x^2+y^2\right)\right\}\right]$$

$$+2\sum_{n=0}^{\infty}\left(\frac{4\Omega^2}{\beta g}\right)^{\frac{3}{2}}\frac{a^3}{(2nh-z)^3}\left\{\frac{(B_1)_{\sigma=0}}{3}\left(1-\frac{24a^2\Omega^2}{5\beta g(2nh-z)^2}\right)\right.$$

$$\left.+\frac{(B_2)_{\sigma=0}}{15}\left(1-\frac{24a^2\Omega^2}{n\beta g(2nh-z)^2}\right)\right\}+2\sum_{n=1}^{\infty}\left(\frac{4\Omega^2}{\beta g}\right)^{\frac{3}{2}}\frac{a^3}{(2nh+z)^3}$$

$$\times\left\{\frac{(B_1)_{\sigma=0}}{3}\left(1-\frac{24a^2\Omega^2}{5\beta g(2nh+z)^2}+\frac{(B_2)_{\sigma=0}}{15}\left(1-\frac{24a^2\Omega^2}{7\beta g(2nh+z)^2}\right)\right)\right\}$$

$$-(x^2+y^2)ya^5\left(\frac{4\Omega^2}{\beta g}\right)^{\frac{5}{2}}\left(B_1+\frac{B_2}{5}\right)_{\sigma=0}\left\{\sum_{n=1}^{\infty}\frac{1}{(2nh-z)^5}+\frac{1}{(2nh+z)^5}\right\},$$

and

$$v \to \frac{\sqrt{\pi}}{2}xy\left(\frac{a}{z}\right)^3\left(\frac{4\Omega^2}{\beta g}\right)^{\frac{3}{2}}\left[\sum_{n=0}^{\infty}(B_1)_{\sigma=0}\left(-\frac{a^2\Omega^2}{\beta g z^2}\right)^n {}_2F_1'\left(-n, \frac{-\frac{3}{2}-n}{2}, x^2+y^2\right)\frac{\Gamma(2n+3)}{n!\,\Gamma(n+\frac{5}{2})}\right.$$

$$\left.+\sum_{n=0}^{\infty}\frac{(B_2)_{\sigma=0}}{2}\frac{\Gamma(2n+3)}{n!\,\Gamma(n+\frac{7}{2})}\left(-\frac{a^2\Omega^2}{\beta g z^2}\right)^n {}_2F_1'\left(-n, \frac{-\frac{5}{2}-n}{2}, x^2+y^2\right)\right]$$

$$-2\left(\frac{4\Omega^2}{\beta g}\right)^{\frac{5}{2}}xy\left(B_1+\frac{B_2}{5}\right)_{\sigma=0}\left\{\sum_{n=1}^{\infty}\frac{1}{(2nh-z)^5}+\frac{1}{(2nh+z)^5}\right\},\quad w\to 0.$$

Thus we see that as the frequency of oscillation approaches zero the steady-state solution exists. The above expressions for u and v show that the limiting velocity is no longer independent of z but varies in a direction parallel to the rotation axis. Thus, due to the stable stratification present in the field, there is vertical shear in the horizontal velocity, resulting in the violation of the Taylor–Proudman constraint which holds in the case of homogeneous rotating fluid.

The forces acting on the body are calculated in this case up to the $O(h^{-5}, \lambda^2)$ and it is found that

$$F_x = 0;\quad F_z = \frac{2\pi\rho_0' a^3\lambda}{3}\left(\frac{\beta a^2\Omega^2}{5}-g\right);$$

$$F_y = 4\rho_0' a^3 U\Omega\lambda\left\{\frac{2\pi}{3}-\frac{4\lambda}{3}\left(\frac{\beta g}{4r^2}\right)^{\frac{1}{2}}+\frac{11}{36}\frac{\zeta(5)}{h^5}\left(\frac{4\Omega^2}{\beta g}\right)^{\frac{5}{2}}+\frac{8\lambda\zeta(3)}{3\pi h^3}\left(\frac{u\Omega^2}{\beta g}\right)-\frac{34\lambda\zeta(5)\times 16\Omega y}{45\pi h^5\beta^2 g^2}\right\}.$$

Thus we conclude that it is possible to obtain the steady-state solutions on the basis of linearized theory when a body is moving in a non-homogenous rotating medium confined partially by the external boundaries. Also from the expression of the transverse force, we see that the presence of the plane horizontal boundaries increases the resistance in a direction perpendicular to the uniform motion of the body.

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